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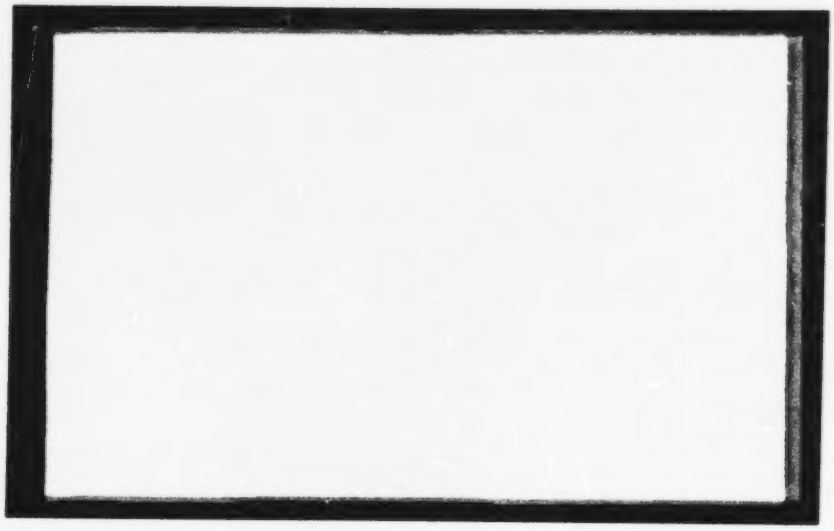

CENTRE DE RECHERCHE SUR LES TRANSPORTS

CASE POSTALE 6128, SUCCURSALE A, MONTRÉAL, QUÉBEC, H3C 3J7

ESTIMATING ORIGIN-DESTINATION MATRICES FROM OBSERVED FLOWS

by

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This paper was written to serve as notes for the First Course on Transportation Planning Models of the International School of Transport Planning given at Amalfi, Italy, October 11-16, 1982. The underlying research was supported in part by a grant from the National Sciences and Engineering Research Council Canada.

Cette recherche a été publiée grâce à une subvention du Fonds F.C.A.C. pour l'aide et le soutien à la recherche.

Université de Montréal

Centre de recherche sur les transports - Publication #263
Département d'informatique et de
recherche opérationnelle - Publication #449

Août 1982

Dépôt légal - 3e trimestre 1982

ABSTRACT

The high cost of surveys, combined with the trend in transportation planning towards a shift from long term planning studies to intermediate and short term planning with limited budgets, have led in recent years to the development of low cost techniques for estimating trip matrices from network data. This paper provides a comprehensive review of current models and methods for estimating trip matrices from traffic counts. After introducing a unifying framework, we review both the regression-based models and the mathematical programming-based models. For this latter subgroup the emphasis is placed on the maximum entropy formalism.

RÉSUMÉ

Le coût souvent élevé des enquêtes origine-destination et la tendance actuelle en planification en transport de ne conduire que des études à court terme ne requiérant que des ressources limitées, ont mené récemment au développement des techniques efficaces pour la reconstruction des matrices de déplacements origine-destination à partir des comptages de trafic. Ce document est une revue critique des modèles et des méthodes courantes de reconstruction de matrices origine-destination. Cette revue s'étend aussi bien sur les modèles de régression que sur ceux de programmation mathématiques. Pour ce dernier groupe, l'accent est mis sur l'approche de maximisation de l'entropie.

THE DEMAND FOR TRAVEL between origin and destination points in a network is usually structured into the form of a two dimensional matrix, called an origin-destination matrix. This is an essential ingredient in a wide variety of travel analysis and planning studies. For example, in urban transportation planning, trip matrices intervene in both the design and assessment of alternative traffic management schemes. At the design level, trip matrices for different transportation modes and different time periods are essential for defining planning options and drawing up effective tactical and operational schemes. Defining bus routes, fixing bus line frequencies, setting one-way streets, reorganizing congested intersections, scheduling snow removal and repairs on urban street networks are examples of such schemes. To assess the impact of alternative traffic management schemes, traffic assignment models are used to determine variations in link flows on the network and trip matrices are part of the basic input to these models.

Traditionally, the estimation of trip matrices involves origin-destination surveys. If reasonable accuracy is to be expected from the resulting estimates, a relative large sample size should be considered. Even for small size samples (1-5%) origin-destination surveys are expensive in terms of manpower requirements, time consuming and difficult to conduct.

Aerial photography is a novel alternative to origin-destination surveys for deriving estimates of trip matrices for small size areas, but this is to a large extent an experimental method at present. Needless to say that it is also currently highly expensive due to the use of technologically advanced photographic equipment and airplanes, and the necessity for a great deal of data processing. See Desforges (12) for a practical application.

The high cost of surveys, combined with the trend in transportation planning towards a shift from strategic long term planning studies involving huge budgets to tactical, intermadiate and operational short term planning with limited budgets, have led in recent years to the development of low cost techniques for estimating trip matrices from partial information. In particular, much work has been done on gravity distribution models which led essentially to methods of estimating trip matrices from estimates of marginal totals. More recently, models for estimating trip matrices from traffic counts have been developed. Marginal totals are more easily and accurately estimated than the trip matrix itself, while traffic counts are relatively inexpensive to collect, and indeed are routinely collected for other purposes, such as accident analysis and network repairs scheduling.

This paper provides a comprehensive review of current models for estimating trip matrices from traffic counts. The emphasis is on person-trips by private automobiles on urban and interurban networks. In general, the reviewed methodology is also applicable to other commodities and networks.

We review both the modelling and the algorithmic development for these models. Existing applications are mentioned but not detailed.

First, we introduce a general framework and describe two basic formulations, one associated with regression-based models and the other with mathematical programming-based models. For this latter subgroup, emphasis is placed on the maximum entropy formalism. Next, we devote our attention to linear and non-linear regression models. The mathematical programming-based models are the subject of the three following

sections. These models are divided into models for single route networks and general networks. Models for single route networks are essentially standard biproportional distribution models. Models for general networks are again divided into multiproportional and network equilibrium-based models. A summary completes this document.

A GENERAL FRAMEWORK

Consider a transport network modeled by an abstract graph defined by a set N of nodes and a set A of directed arcs. Some nodes $i \in N$ are zone origins and/or destinations, others are transhipment nodes.

The traffic demand between origin node i and destination node j is denoted by T_{ij}, and let T denote the matrix $\{T_{ij}\}$. The distribution of $\{T_{ij}\}$ onto the routes r of the set R of all routes between i and j induces the traffic flows v_a on all arcs $a \in A$. Let $s_a(v_a)$ be the cost function of arc $a \in A$ an increasing function of v_a. $s_a(v_a)$ may simply be a delay function (travel time) or a generalized cost function taking into account time, distance and out-of-pocket cost. The cost of route r is the sum of costs incurred on all arcs forming r

$$u_r = \sum_{a \in A} \delta_{ar} \, s_a(v_a) \tag{1}$$

with

$$\delta_{ar} = 1 \text{ if a is an arc of r, 0 otherwise} \tag{2}$$

Within this abstract setting, the central problem of interest is: given a reference trip matrix $t = \{t_{ij}\}$ and a set of observed arc flows (traffic counts) determine the best trip matrix T consistent with the observed arc flows. t may be an initial estimate of T, in which case t is a target matrix, or it may be an old matrix giving information on the a priori cell probabilities, or finally simply a matrix composed of 1 or 0 cell values which fix the structural zeros in T.

To be able to describe the intimate relationship between the traffic demand T and the induced arc flows we must first examine the process of allocating the traffic demand to network routes, commonly referred to as traffic assignment.

In equilibrium assignment, the allocation of traffic demand onto routes obeys Wardrop's (62) first principle and gives rise to the following relations

$$\sum_{r \in R^*_{ij}} n_{ijr} = 1 \tag{3}$$

$$n_{ijr} = 0, \quad \text{for } r \notin R^*_{ij} \tag{4}$$

where n_{ijr} is the proportion of demand T_{ij} using route r, and R^*_{ij} is the set of routes which have interzonal costs equal to the minimum value

$$u_{ij} = \min_{r \in R_{ij}} u_{ijr} \tag{5}$$

Unless the arc costs $s_a(v_a)$ are constants the proportions n_{ijr} and the minimum route costs u_{ij} are a function of the traffic demand. Equilibrium assignment is usually applied to congested networks.

In proportional assignment, n_{ijr} is proportional to the route cost which does not depend on the traffic demand. Typical examples are, all-or-nothing assignment in which n_{ijr} is equal to 1 for one value of r and

0 otherwise, or Dial's (14) multipath assignment which distributes the demands onto routes according to a logit model

$$n_{ijr} = \exp(-\theta u_{ijr}) / \sum_{k \in R_{ij}} \exp(-\theta u_{ijk}) \tag{6}$$

where θ is a calibrated parameter. Proportional assignment is most suited for uncongested networks.

The arc flows $\{v_a\}_A$ resulting from an assigment are related to the traffic demand T through the fundamental relation

$$v_a = \sum_{(ij) \in W} n_{ija} T_{ij} \tag{7}$$

with

$$n_{ija} = \sum_{r \in R_{ij}} \delta_{ar} n_{ijr} \tag{8}$$

where W is the set of positive T_{ij}.

When the demand T_{ij} is specified by a demand function

$$T_{ij} = D(X_i, X_j, \beta, C_{ij}) \tag{9}$$

where X_i, X_j are explanatory variables, β is a vector of calibrated parameters and C_{ij} is a measure of the separation between i and j, then the traffic assignment mapping (7) may be rewritten

$$v_a = \sum_{(ij) \in W} n_{ija} D(X_i, X_j, \beta, C_{ij}) \tag{10}$$

In traffic assignment, (7) or (10) is solved to obtain the traffic pattern $\{v_a\}_A$. In this paper, we address the reverse problem which consists of choosing a trip matrix T or a set of parameters β satisfying (7) or (10) for a given subset of observed arc flows $\{\hat{v}_a\}_L$, $L \subseteq A$.

Estimating directly the trip matrix T or calibrating the demand function $D(X_i, X_j, \beta, C_{ij})$ lead to the two following formulations

(P1) minimize $F(T, t)$

subject to

$$\sum_{(ij) \in W} n_{ija} T_{ij} = \hat{v}_a \,, \quad a \in L \tag{11}$$

$$n_{ija} = f(T) \tag{12}$$

$$T \in \Gamma \tag{13}$$

and

(P2) minimize $F(\beta) = \sum_{a \in L} \varepsilon_a^2$

subject to

$$\varepsilon_a + \sum_{(ij) \in W} n_{ija} D(X_i, X_j, \beta, C_{ij}) = \hat{v}_a, \quad a \in L \tag{14}$$

$$n_{ija} = g(\beta) \tag{15}$$

$$C_{ij} = h(\beta) \tag{16}$$

where the ε_a are random errors assumed independently and identically distributed with null mean and finite variance.

Constraint (13) will allow the model to take into account any additional information on the structure of the trip matrix T or relevant cost information such as information on the total number of trips, or on

the mean trip cost, etc. Equations (12), (15) and (16) express the general case where the traffic proportions n_{ija} and the interzonal travel costs C_{ij} are implicit functions of the traffic demand.

When the n_{ija} and the C_{ij} are independent of the traffic demand and exogenously estimated, problem (P2) reduces to a least squares regression problem with appropriate assumptions on the residual terms.

For the direct estimation of the trip matrix T, the objective function F(T,t) may be any suitable distance metric between T and the reference matrix t. Standard possibilities for the objective function F are (Smith and Hutchinson (55))

i) $$F(T,t) = \sum_{(ij)\varepsilon W} \omega_{ij} |T_{ij} - t_{ij}| \qquad (17)$$

ii) $$F(T,t) = \max_{(ij)\varepsilon W} \{\omega_{ij} |T_{ij} - t_{ij}|\} \qquad (18)$$

iii) $$F(T,t) = \sum_{(ij)\varepsilon W} \omega_{ij} (T_{ij} - t_{ij})^2 \qquad (19)$$

iv) $$F(T,t) = \sum_{(ij)\varepsilon W} \omega_{ij} T_{ij} \text{Log}(T_{ij}/t_{ij}) \qquad (20)$$

(17) to (19) are the usual weighted L_1, L_∞ and L_2 norms. With linear constraints problem (P1) with the objective function given by (17) or (18) leads to a linear program. More commonly found in the literature are the quadratic objective function (19) and the convex entropy function (20). In the case where no a priori information is available ($t_{ij} = 1$, $(ij)\varepsilon W$) only the entropy function may be used, and for this reason it is usually preferred over the other criteria.

The objective function (20) can be derived by several distinct considerations. For the models in this paper we will adhere to the maximum entropy formalism. This formalism stems from Jaynes (31) who proposed that Shannon's measure of uncertainty (Shannon (53)) be used to define values for porbability distributions in problems of statistical inference. In particular, if the given information is in the form

$$\sum_i p_i = 1 \qquad (21)$$

$$\sum_i p_i f_j (x_i) = a_j, \; j=1,2,\ldots \qquad (22)$$

the least presumtive ("maximally non-committal") probability distribution is the set of p_i which maximizes

$$E = -K \sum_i p_i \log p_i \qquad (23)$$

subject to (21) - (22),
where K is a scaling factor. Any other choice "would amount to arbitrary assumptions of information, which by hypothesis, we do not have" Jaynes (31). This method was introduced to transportation and regional analysis by Wilson (67, 68). Extensions towards using an a priori distribution $\{q_i\}$ have been suggested by Kullback (35) and Snickars and Weibull (57), leading to the function

$$E = -K \sum_i p_i \log (p_i/q_i) \qquad (24)$$

Another major issue in the formulation of (P1) is that inconsistency in the observed flows mainly due to measurement errors and temporal

variability may lead to an infeasible system of constraints. Two distinct approaches may be taken to deal with inconsistency in the data in order to ensure the existence of a feasible solution to problem (P1).

A first approach consists in estimating a new set of consistent flows $\{\vartheta_a\}_L$ from the observed flows $\{\hat{\vartheta}_a\}_L$. For example, the estimation may be carried out by solving a least squares deviation problem

$$\text{Min} \sum_{a \in L} (\hat{\vartheta}_a - \vartheta_a)^2 \tag{25}$$

$$\text{subject to} \quad Ey = 0 \tag{26}$$

$$y \geq 0 \tag{27}$$

where E is the circulation flow node-arc incidence matrix of the modeling graph. A maximum likelihood estimation under the assumption that the observed flows are Poisson distributed with mean ϑ_a, has been suggested in Van Zuylen and Willumsen (60).

A second approach consists of replacing each equation (11) by an interval constraint

$$\hat{\vartheta}_a - \varepsilon\hat{\vartheta}_a \leq \sum_{(ij) \in W} n_{ija} T_{ij} \leq \hat{\vartheta}_a + \varepsilon\hat{\vartheta}_a$$

where ε is a user specified tolerance fraction. Such an approach has been used in Tremblay (59) and Dugas (15).

Having described the general modeling approach for the estimation of origin-destination trip matrices, the next sections will review current models and methods. We will first examine the regression-based models.

LINEAR AND NON-LINEAR REGRESSION MODELS

The first attempt to use traffic counts to calibrate a gravity type demand model was apparently made by a group of City of Montreal traffic planners (Arbour et al. (1), (2), (3)) as reported in Gaudry et al (21). The purpose of this work was to estimate the ridership of the proposed new subway lines. This unpublicised work and all subsequent contributions until 1977, are centered on the specification and calibration of linear and non-linear demand models.

Linear models have been suggested by Low (41), Overgaard (50), Symon et al. (58), Holm et al. (29) and more recently by Lamarre (36) (see also Gaudry and Lamarre (21)). None of the proposed models but Holm et al's consider the dependency of the traffic proportion n_{ija} on the demand to be estimated and all-or-nothing assignment is always assumed.

The general functional form of the demand advanced by Low, Overgaard, Symon et al. and Holm et al. is

$$T_{ij} = \beta_o + \sum_m \beta_m X_{ijm} , \tag{28}$$

with the traffic X_{ijm} from i to j for purpose m taking a gravity type product form

$$X_{ijm} = f_m (A_{im}, B_{jm}, C_{ij}) \tag{29}$$

where A_{im}, B_{im} are explanatory variables such as population and employment (Low (41)), the percentage of single family households (Overgaard (50)), a recreational attractiveness index (Symon et al. (58)).

Only Gaudry and Lamarre (21) specified a simpler model

$$T_{ij} = \beta_o + \beta_1 (A_i + B_j) + \beta_2 C_{ij} \qquad (30)$$

In general, linear models have been found to produce less realistic results than non-linear models (see for example Lamarre (36)).

An interesting feature found in Holm et al.'s model is the use of Smock's (56) assignment procedure to iteratively refine the traffic proportions n_{ija}. This is a first attempt to take into account the dependency of the traffic proportions n_{ija} and the interzonal traveling costs C_{ij} on the demand (equations (15) and (16) in problem (P2)) in a regression model. Holm et al.'s heuristic procedure works essentially as follows. At a general iteration q one has a set $\{n_{ija}^q\}$ and current arc flows $\{x_a^q\}_A$. C_{ij}^q is the minimum route cost from i to j, the arc costs being $s_a(x_a^q)$. First, estimate the parameter of the following function

$$T_{ij} = \beta A_i B_j C_{ij}^{-P} \quad (P \text{ is fixed}) \qquad (31)$$

by solving the regression model (14) with $C_{ij} = C_{ij}^q$ and $n_{ija} = n_{ija}^q$. This provides a new trip matrix T^q. Next, determine the all-or-nothing assignment $\{y_a\}_A$ and $\{n_{ija}\}$ using the trip matrix T^q. Finally, revise the current flows and traffic proportions as follows

$$x_a^{q+1} \leftarrow x_a^q + \frac{1}{q} (y_a - x_a^q) \qquad (32)$$

$$n_{ija}^{q+1} \leftarrow n_{ija}^q + \frac{1}{q} (n_{ija} - n_{ija}^q) \qquad (33)$$

and proceed to iteration (q+1). The procedure terminates when the variation on the value of β from one iteration to the next reaches an acceptable level.

Holm et al. have reported several applications, mainly in rural areas. Since rural networks are generally uncongested, these applications did not benefit from the innovative adjustment scheme described above.

Non-linear regression approaches have been advocated by Wills (64), (65). Robillard (52), Hogberg (28) and Debaille (11). All the demand models proposed are variants of the gravity distribution model, and the n_{ija} are assumed independent of the demand and exogenously determined. Robillard and Debaille proposed a simple double factor form leading to the non-linear regression model

$$\hat{v}_a = \sum_{(ij)\in W} n_{ija} \alpha_i \beta_j f_{ij} + \varepsilon_a \,, \quad a \in L \qquad (34)$$

which may be solved using the Lawton and Sylvestre (38) method. No particular form has been given to the separation factor f_{ij} in Robillard, while Debaille uses the following formula

$$f_{ij} = (t_{ij} \times t_{ji})^{1/2} \left(\pi_k \frac{t_{ki}}{t_{ik}} \times \frac{t_{kj}}{t_{jk}}\right)^{1/2n} \,,$$

$$i,j = 1,2,\ldots,n \qquad (35)$$

where $\{t_{ij}\}$ is a given reference matrix. Note that $f_{ij} = f_{ji}$. This symetry is aimed at permitting a morning peak-hour reference trip matrix to be used in the estimation of the evening peak-hour trip matrix.

Hogberg (28) proposed the use of the Marquardt (43) non-linear

regression method to estimate the parameters β_n, n=1,2,...,5 of the following demand model adapted from Bexelius et al. (7).

$$T_{ij} = f(d_{ij}) \times \left(\beta_1 \frac{A_{i1} A_{j2}}{\sum_k A_{k2} f(d_{ik})} + \beta_2 \frac{A_{i1} A_{j1}}{\sum_k A_{k1} F(d_{ik})} + \beta_3 \frac{A_{i2} A_{j2}}{\sum_k A_{k2} f(d_{ik})}\right) \tag{36}$$

where

$$f(d_{ij}) = d_{ij}^{\beta_4} \exp\left(\beta_5 (\log d_{ij})^2\right) \tag{37}$$

d_{ij} is the distance separating i and j, A_{i1} and A_{i2} are respectively the population and employment in i. The three components in (36) correspond to the three subclasses of home-to-work, home-to-home and work-to-work trips.

Finally, Wills (64), (65) suggested a product form demand model

$$T_{ij} = \beta_o \prod_m (A_{im} B_{jm})^{\beta_m} \tag{38}$$

where A_{im}, B_{jm} are the m^{th} explanatory variables associated with i and j.

Lamarre's (36) and Wills' (64), (65) models have been used in the estimation of intercity trip matrices for the provinces of Quebec and British Columbia. Debaille's (11) model has been extensively tested with data from Roanne, France. The model's estimates, corresponding to different sizes and time periods were compared with trip matrices obtained from aerial photography (Desforges (12)). It was found that the quality of the estimates is highly sensitive to imprecision in the traffic proportions n_{ija}, and that significant improvement in the quality of the estimates might be obtained if the n_{ija} were endogenously determined as stressed in the general formulation (P2).

MATHEMATICAL PROGRAMMING-BASED MODELS AND METHODS

This section focusses on current mathematical programming-based methodologies for estimating trip matrices from observed flows.

First, we assume that traffic proportions n_{ija} are independent of the traffic demand to be estimated and exogenously determined. This assumption is valid for uncongested networks, such as rural or intercity networks with few alternative routes between any given origin-destination pair. For these networks, the traffic proportions may be adequately estimated or obtained from the application of any proportional assignment procedure. With fixed n_{ija}, problem (P1) with the entropy objective function and linear constraints leads to a multiproportional model. For single route networks, such as bipartite transportation networks and linear networks, we obtain the well known biproportional model.

After reviewing the biproportional and the multiproportional models and methods, we will examine models which take into account traffic congestion. The traffic proportions n_{ija} are then assumed consistent with Wardrop's (62) first route choice principle, leading to network equilibrium-based models.

MODELS AND METHODS FOR SINGLE ROUTE NETWORKS: THE BIPROPORTIONAL MODEL - Single route networks are transportation systems or subsystems in

which there is at most one route connecting any origin-destination pair. Typical examples of such networks are one bus line with bus stops or a segment of freeway with entrance and exit ramps.

In general, the traffic counts are collected on all access and egress arcs of such networks, so the system of constraints (11) reduces to the constraints on the production and attraction of zones found in standard transportation problems

$$\sum_j T_{ij} = a_i \tag{39}$$

$$\sum_i T_{ij} = b_j \tag{40}$$

$$T_{ij} \geq 0 \tag{41}$$

$$T_{ij} = 0 , \quad (ij) \notin W \tag{42}$$

Consistency of the observed flows implies the equality

$$\sum_i a_i = \sum_j b_j \tag{43}$$

With the entropy objective function

$$F(T,t) = \sum_{(ij)\in W} T_{ij} \log(T_{ij} / t_{ij}) \tag{44}$$

the optimal solution to problem (P1) is

$$T_{ij} = \alpha_i \beta_j t_{ij} \tag{45}$$

subject to (39) - (42).

This multiplicative form (45) is commonly known as the biproportional model (Bacharach (4)). To obtain the solution to our estimation problem, we may solve the primal problem

$$\min F(T,t) \tag{46}$$

subject to (39) - (42)

with an adaptation of any existing convex programming technique such as the convex-simplex method, the reduced-gradient method etc. (Zangwill (69)). However, there is an effective primal-dual method specially developed to solve (45) directly by alternatively adjusting the row multipliers and the column multipliers as follows

$$\text{(initialisation)} \qquad \beta_j = 1, \text{ all } j$$

$$\text{(row balancing)} \qquad \alpha_i \leftarrow \frac{a_i}{\sum_j t_{ij}\beta_j} \qquad i = 1,2,\ldots \tag{47}$$

$$\text{(column balancing)} \qquad \beta_j \leftarrow \frac{b_j}{\sum_i t_{ij}\alpha_i} \qquad j = 1,2,\ldots \tag{48}$$

This method, called the double factor method, was introduced for the first time in 1937 by Kruithof (34) for the prediction of telephone traffic distributions. Deming and Stephan (13) independently rediscovered the method. They applied it to a cross-classification problem as

a simplification of least squares fitting.

In transportation planning (45) is commonly refered to as the gravity model and t_{ij} may also be replaced by a separation function. The double-factor method is then called the Furness iterative method (Furness (20).

The first proof of convergence of the double factor method has been given by Gorman (23). Since then, many independent convergence proofs have been given, and variants of the method with better convergence and extensions of the model have been studied (Wilson (68); Murchland (44); Robillard and Stewart (51); Evans and Kirby (19): Macgill (42); Jefferson and Scott (32); among others).

The biproportional model and method have also been used in many areas outside of transportation (Wilson (68); Evans (18)) and telecommunication (Bear (6)). In statistics, maximum likelihood estimation of "log-linear" models for cross-classification problems leads directly to biproportional problems (Bishop et al (8); Ireland and Kullback (30)). Biproportional models also appear in the estimation of transition probabilities of doubly stochastic Markov chains (Sinkhorn (54)). In economics, the model arises in the estimation of Leontief input-output matrices (Bacharach (4)).

Since the estimation of a trip matrix, associated with a single route network from observed flows, reduces to the well studied biproportional model, it is not surprising then to find that variants of the biproportional model and method are widely used in the estimation of linear network trip matrices (Eldor (16); Nihan (49); Hauer and Shin (27)). Some sensitivity analyses of the biproportional model are found in Eldor (16) and Willis and May (63). It was found, that in spite of some discrepancies between the estimates and the survey-based trip matrices, the system-wide measure of effectiveness was practically insensitive to these discrepancies.

METHODS AND MODELS FOR GENERAL NETWORKS: THE MULTIPROPORTIONAL MODELS - This section describes two intimately related maximum entropy models for the estimation of trip matrices from observed flows. The traffic proportions are still assumed fixed exogenously. For notational ease, we shall from now on use index i to designate the i^{th} origin-destination pair.

Let z_{ia} be the flow on arc a which has been produced by the traffic demand T_i, $i \in W$. Since the traffic proportions n_{ia} are given, the T_i, $i \in W$ may be derived from estimates of z_{ia}, $(ia) \in S$ with

$$S = \{(i,a) \in W \times L \mid n_{ia} > 0\} \tag{49}$$

Define p_{ia} as the probability that an observed unit of flow has been produced by a trip associated with the i^{th} origin-destination pair on arc a. This probability is

$$p_{ia} = z_{ia} \Big/ \sum_{a \in L} v_a \tag{50}$$

and the corresponding a priori probability computed from the a priori information $\{t_i\}_w$ is

$$q_{ia} = t_i \, n_{ia} \Big/ \sum_{(ia) \in S} t_i \, n_{ia} \tag{51}$$

Using Jaynes' maximum entropy formalism the least biased probability distribution $\{p_{ia}\}_S$ is that distribution which maximizes the entropy function subject to any given information. This leads to the following model

$$\max \; - \sum_{(ia)\in S} p_{ia} \log(p_{ia}/q_{ia}) \tag{52}$$

subject to

$$\sum_{(ia)\in S} p_{ia} = 1 \tag{53}$$

$$\sum_{(ie)\in S} p_{ie} f_a(z_{ie}) = \hat{v}_a / \sum_{a\in L} \hat{v}_a \;, \quad a\in L \tag{54}$$

$$p_{ia} \geq 0 \tag{55}$$

where

$$f_a(z_{ie}) = \begin{cases} 1 & \text{if } a=e \\ 0 & \text{otherwise} \end{cases} \tag{56}$$

Substituting $T_i\, n_{ia}$ for z_{ia} in (50), eliminating constant terms from the objective function and deleting the redundant equation (53), one obtains the following weighted entropy problem in variables T_i, $i\in W$

$$\max \; - \sum_{i\in W} K_i\, T_i \log (T_i/t_i) \tag{57}$$

subject to

$$\sum_{i\in W} T_i\, n_{ia} = \hat{v}_a \;, \quad a\in L \tag{58}$$

$$T_i \geq 0 \;, \quad T_i = 0 \;, \quad i\notin W \tag{59}$$

where

$$K_i = \sum_{a\in L} n_{ia} \tag{60}$$

The solution to this can be expressed in the form of a multiproportional model (see for example Murchland (44))

$$T_i = t_i \prod_{a\in L} \alpha_a^{\,n_{ia}/K_i} \;, \quad i\in W \tag{61}$$

subject to (58)

where

$$\alpha_a = e^{-1/n_{ia}|L|}\, e^{-\lambda_a} \tag{62}$$

and λ_a are the Lagrangian multipliers.

The original idea of deriving a trip matrix T from the least biased estimates of the number of arc trips ($T_i\, n_{ia}$) stems from Zuylen in Zuylen and Willumsen (60). Zuylen used a different argument based on Brillouin's (10) information measure to derive the same multiproportional model (61).

We now examine a related approach for directly estimating the most likely trip distribution $\{T_i\}_W$ under assumed additional information on the total number of trips $M = \sum_{i\in W} t_i$. Suppose that M is given, define

p_i as the probability of a trip associated with the i^{th} origin-destination pair ($p_i = T_i/M$). Using again the maximum entropy formalism, the most likely probability distribution is that set of p_i which solves

$$\max \; - \sum_i p_i \log (p_i/q_i) \tag{63}$$

subject to

$$\sum_{i \in W} p_i = 1 \tag{64}$$

$$\sum_{i \in W} p_i \, n_{ia} = \hat{v}_a/M \,, \quad a \in L \tag{65}$$

$$p_i \geq 0 \quad \text{and} \quad p_i = 0 \quad \text{if } q_i = 0 \tag{66}$$

where the a priori probability q_i is

$$q_i = t_i \Big/ \sum_{i \in W} t_i \tag{67}$$

The solution of this in variables T_i is

$$T_i = t_i \, \alpha_o \prod_{a \in L} \alpha_a^{n_{ia}} \tag{68}$$

subject to (64) - (65)

where

$$\alpha_o = e^{-\lambda_o - 1} \, M \Big/ \sum_{i \in W} t_i \tag{69}$$

$$\alpha_a = e^{-\lambda_a} \tag{70}$$

and λ_o, λ_a are Lagrangian multipliers. The difference in the exponent of the multipliers with respect to the first model, reflects the fact that we maximize the entropy of two distinct distributions, $\{T_i \, n_{ia}\}_S$ in the first model and $\{T_i\}_W$ in the second. The multiproportional model (68), has been obtained by Willumsen in Zuylen and Willumsen (60). His derivation is based on Wilson's approach (Wilson (67)) and rather than imposing explicitly constraint (64) Willumsen assumed that (65) would enforce (64). In other words, all trip matrices which reproduce the set of observed flows $\{\hat{v}_a\}_L$ are assumed to have a fixed unknown total number of trips. It is unlikely that this assumption would be verified in practice. However, since the total number of trips is not known exactly, it can be argued that relaxing constraint (64) may not after all be worse than using a bad estimate of the true value of M.

When the proportions n_{ia} take values 0 or 1 only (all-or-nothing assignment) solving (61) or (68) consists simply of normalizing each constraint in turn cyclically until convergence (Murchland (44)). However, for general values $0 \leq n_{ia} \leq 1$, the former effective balancing scheme may not converge at all, and another method should be used. As noted earlier the primal formulation (61) or (68) is a strictly convex program with linear constraints so that any existing primal convex programming technique may be used. In practice though, the large size and low density of the coefficient matrix $\{n_{ia}\}$ leads to the consideration of efficient iterative row-generation techniques in which only one

constraint is considered at a time. We will review two such methods. The first one, by Bregman (9), is a non-orthogonal projection method. Kruithof's double-factor method, Evans and Kirby's triproportional method and Murchland's multiproportional balancing procedure are special cases of this method for binary coefficient matrices (see Lamond and Stewart (37)). The second method, also a primal-dual method, has been developed by Gordon et al. (22) for algebraic image reconstructions in radiology.

Consider the general problem

$$\min \sum_j x_j \, (\log x_j/y_j - 1) \tag{71}$$

subject to

$$\sum_j a_{ij} x_j = b_i \qquad i=1,2,\ldots,n \tag{72}$$

$$x_j \geq 0 \qquad j=1,2,\ldots,n \tag{73}$$

where $b_i > 0$, $\forall_i$, and $y_j > 0$, $\forall_j$. 0 log 0 is defined to be 0. We assume that an optimal solution x exists and therefore is unique

Bregman's algorithm to solve the above problem is

Initialisation $x_j^o = y_j$, $j=1,2,\ldots,n$

$q = 0$ (index of iteration)

$i = 1$

General step (balancing the i^{th} equation)

determine the unique solution x^{q+1} and λ of

$$x_j^{q+1} = x_j^q \exp(-\lambda a_{ij}), \; j=1,2,\ldots,n \tag{74}$$

and

$$\sum_j a_{ij} x_j^{q+1} = b_i \tag{75}$$

$q \leftarrow q + 1$

$i \leftarrow (i \text{ module } q) + 1$

Convergence of this algorithm to the optimal solution x* was proved by Bregman under the assumption that if $x^q > 0$ then x^{q+1} defined uniquely by (74) - (75) is also > 0. Bregman also generalised the above algorithm to handle inequality constraints as well.

Note that if the a_{ij} take the values 0 or 1 only, equation (75) is linear in exp(-λ) and Bregman's algorithm reduces to the multiproportional iterative adjustment method. One may also interpret Bregman's algorithm as a primal-dual method where at each iteration only one dual variable is modified and its variation is equal to -λ.

The second method known under the acronym MART (multiplicative algebraic reconstruction technique) may also be used to solve problem (71) - ((73). Just as in the ordinary balancing method, the current solution x^q is adjusted iteratively using each constraint in turn

cyclically as follows

$$x_j^{q+1} = \left(\frac{b_i}{\sum_j a_{ij} x_j^q}\right)^{a_{ij}} x_j^q \quad , \quad j=1,2,\ldots,n \qquad (76)$$

The convergence of MART is proved by Lent (40) under the assumption that $0 \leq a_{ij} \leq 1$.

Willumsen and Van Vliet (61) have tested Willumsen's model with data for the central district of Reading, England. It was found that the distance between the trip matrix estimated from traffic counts and the observed trip matrix is of the same order as the distances between matrices observed at different time periods. Variants of the multiproportional models considered here have been used by Tremblay (59) to estimate telephone traffic on a Canadian Bell subnetwork.

For applications, in which the all-or-nothing assignment is accepted as a reasonable allocation of traffic demand in the network then a multiproportional model, with its simple structure, and a minimal input data requirements seems to be a good choice, unless a behavioural demand model with some predictive capabilities is needed.

For congested networks where multiple routes are utilized between any given origin-destination pair, there is no known method for accurately estimating the traffic proportions n_{ia} separately from the traffic demand T_i. This motivated the development of models which obviate the need to carry out a prior estimation of the traffic proportions n_{ia}, by integrating trip demand estimation and network equilibrium traffic assignment into one process. Nguyen (45), (46), (48), Babin (5), Jornsten and Nguyen (33), Erlander, Nguyen and Stewart (17), Gur et al. (24) and LeBlanc and Farkanglan (39) have contributed to the development of the methodology which is described in the following section.

INTEGRATED TRIP DEMAND ESTIMATION AND EQUILIBRIUM TRAFFIC ASSIGNMENT MODELS - The key assumption underpinning the modeling approach described in this section is that the traffic pattern from which the set of arc flows $\{\vartheta_a\}_L$ has been observed is an equilibrium traffic pattern consistent with Wardrop's first route choice principle (equations (3) - (5)). We further assume that the total number of trips M is given, so that the maximum-entropy formalism may be used to generate the most likely trip distribution $\{T_i\}_W$ among a certain set of candidates. The definition of this set of candidates (which is a substantial departure from previous approaches) will now be analysed. We consider the case where a complete set of observed arc flows $\{\vartheta_a\}_A$ is given, so that it is possible to compute the minimum route costs $\{\hat{u}_i\}_W$ from the arc costs $S_a(\vartheta_a)$. Alternatively, the route costs $\hat{u}_i$ may also be estimated directly without recourse to the arc flows.

Next, we analyse the case where only a subset $\{\vartheta_a\}_L$, $L \subset A$ of observed flows is provided.

COMPLETE SET OF OBSERVED ARC FLOWS - For a given demand $\hat{T}$, the associated equilibrium traffic pattern $\{\vartheta_a\}_A$ is uniquely defined for increasing functions $s_a(v_a)$ (see for example Nguyen (47)) and produces a unique set of intersonal minimum route costs $\{\hat{u}_i\}_W$ such that

$$\hat{u}_i = \min_{r \in R_i} \hat{u}_{ir} , \qquad i \in W \tag{77}$$

$$\hat{h}_{ir} = 0 , \quad \text{if } r \notin \hat{R}_i \tag{78}$$

and

$$\hat{R}_i = \{r \mid \hat{u}_{ir} = \hat{u}_i\} . \tag{79}$$

The question of interest is: given the set $\{\hat{u}_i\}_W$ and the knowledge that $\{\hat{u}_i\}_W$ satisfies (77) - (79) can we reproduce the equilibrium traffic pattern $\{\hat{v}_a\}$ by allocating some traffic demand T onto the network? The answer is "yes", as shown by Nguyen (45) through the following simple model

$$\text{Min } f(V,T) = \sum_a \int_0^{v_a} s_a(x)\, dx - \sum_{i \in W} \hat{u}_i T_i \tag{80}$$

subject to

$$T_i - \sum_{r \in R_i} h_{ir} = 0 \qquad i \in W \tag{81}$$

$$h_{ir} \geq 0 \qquad r \in R_i , \; i \in W$$

$$T_i \geq 0 , \; T_i = 0 \quad i \notin W \tag{82}$$

where h_{ir} is the flow on route r connecting the i^{th} origin-destination pair.

In the following, results are stated without proof. Proofs are given in Nguyen (45), (48). Problem (80) - (82) is strictly convex in variables $\{v_a\}_A$ and linear in variables $\{T_i\}_W$ with linear constraints, it always has a solution (V,T), and further more there is a unique optimal solution $\{v_a^*\}_A$. Any optimal solution (V^*,T^*) has the following properties

- $\{v_a^*\}_A$ is an equilibrium traffic pattern,
- for all $T_i^* > 0$, the minimum route cost u_i^* is equal to $\hat{u}_i$, $i \in W$.

Since $\{\hat{v}_a\}_A$ and $\{\hat{u}_i\}_W$ satisfy the Kuhn and Tucker optimality conditions for problem (80) - (82), we obtain that any optimal solution $\{V^*,T^*\}$ reproduces the observed traffic pattern $\{\hat{v}_a\}_A$.

We have thus found an alternative way of generating a trip matrix consistent with the observed flows without referring to constraints (11). This avoids the difficult estimation of the traffic proportions $\hat{n}_{ia}$. In practice, the closeness of the estimate to the observed flows $\{\hat{v}_a\}_A$ depends, partly on errors in input data (such as measurement errors inherent in the observed flows, network errors introduced in coding and agregation, in cost function calibrations, etc.) and partly on the realism of the traffic equilibrium assumption.

Let Ω be the set of all optimal trip matrices of problem (80) - (82). Then, the most likely trip matrix consistent with the observed flows $\{\hat{v}_a\}_A$ is that trip matrix which solves

$$\text{Max} \quad -\sum_i T_i \log (T_i / t_i) \tag{83}$$

subject to

$$\sum_{i \in W} T_i = M \tag{84}$$

$$T \in \Omega \tag{85}$$

Since the set of candidates Ω is not known explicitly, an immediate approach for solving (83) - (85) would be to approximate Ω by a polyhedron defined by a finite number of linear constraints. This approach has been adopted in Jornsten and Nguyen (33) and Nguyen (48). The resulting computational scheme consists essentially of postoptimizing at each iteration a multiproportional problem and an equilibrium traffic assignment problem. More precisely at a general iteration q the steps are

Step 1 Solve the multiproportional problem

$$\text{Max} \quad -\sum_{i \in W} T_i \log (T_i/t_i)$$

subject to

$$\sum_{i \in W} T_i = M$$

$$\sum_{i \in W} \hat{u}_i T_i = C_o$$

$$\sum_{i \in W} u_i^{(k)} T_i \leq C_k \qquad k=1,2,\ldots,q$$

$$T_i \geq 0 , \quad T_i = 0 \text{ if } t_i = 0$$

for $T^{(q)}$.

Step 2 Determine the equilibrium traffic pattern $V^{(q)}$ produced by the traffic demand $T^{(q)}$ and calculate the new constraint

$$\sum_{i \in W} u_i^{(q+1)} T_i \leq C_{q+1}$$

where

$$C_o = \sum_{a \in A} s_a (v_a^{(o)}) \, v_a^{(o)}$$

and

$$C_q = \sum_{a \in A} s_a (v_a^{(q)}) \, v_a^{(o)}$$

$v^{(o)}$ is the optimal solution of (80) - (82) and $u_i^{(q)}$, $i \in W$ are the minimum route costs associated with the traffic pattern $v^{(q)}$.

Techniques for solving the multiproportional problem have been discussed earlier. Several efficient algorithms for the fixed and variable demand equilibrium assignment problems may be used for the traffic assignment in step 2. (Nguyen (47)). Finally, extensions of the above model to handle general asymetric arc cost function $s_a (v_1, v_2, \ldots, v_n)$ have been investigated in Nguyen (48).

Gur et al (24) have implemented a variant of the above model. The main difference is in the objective function. Rather than using the maximum entropy objective function Gur et al. proposed the least squares objective function

$$\min \sum_{i \in W} (T_i - t_i)^2$$

where the "target" matrix $\{t_i\}$ is obtained from the calibration of a particular gravity type distribution model.

Gur et al's (25) resulting software package LINKOD has been extensively tested by Hans et al. (26) using the San Jose CBD network data. It was found that the LINKOD package had some difficulties in estimating very sparse trip matrices, although the traffic flows are well reproduced.

INCOMPLETE SET OF OBSERVED ARC FLOWS - When the data consists only of a subset of observed arc flows one of the two following approaches may be used.

In the data preprocessing phase to remove inconsistencies in the observed flows $\{\hat{v}_a\}_L$, a complete set of feasible flows $\{\hat{y}_a\}_A$ is obtained where the subset $\{\hat{y}_a\}_L$ is the best estimate of the set $\{\hat{v}_a\}_L$ according to the difference metric chosen (19) or (20). Using the set $\{\hat{y}_a\}_A$ to approximate the unknown set $\{\hat{v}_a\}_A$ provides a good method of handling the problem of incomplete traffic counts.

Another approach would be to refine iteratively the unknown equilibrium traffic proportions $n_{ia}(T)$. A possible heuristic scheme similar in spirit to Holm et al's method is

<u>initialisation</u> Determine $n_{ia} = n_{ia}(t)$

$a \in L$, $q \leftarrow 1$ (iteration index)

<u>multiproportional step</u> Find $T^{(q)}$

$$\text{maximizing} - \sum_i T_i \log (T_i/t_i)$$

$$\text{subject to} \sum_{i \in W} T_i = M$$

$$\sum_{i \in W} n_{ia}^{(q)} T_i = \hat{v}_a , \quad a \in L$$

$$T_i \geq 0, \quad T_i = 0 \text{ if } t_i = 0$$

<u>traffic assignment step</u> Determine $n_{ia}^{(q+1)} = n_{ia}(T^q)$, $a \in L$

set $q \leftarrow q+1$

where $n_{ia}(T)$ is obtained from any fixed demand equilibrium traffic assignment procedure, the demand being T.

Conditions for convergence, extensions and refinements of the above scheme are subjects of current research.

SUMMARY

This paper has reviewed a number of the most promissing existing models and methods for estimating trip matrices from observed flows.

Two distinct broad modeling approaches have been analysed; the regression-based and mathematical programming-based approaches. The latter approach stems from the maximum entropy formalism.

Linear and non-linear regression models are most suited to city wide or intercity networks where the propensity to travel can be captured by a variety of socio-economic variables which can be injected into the demand models. At the present time, applications of regression-based models are limited to uncongested networks for which the traffic proportions needed to relate traffic demand to network flows and the interzonal travel disutilities may be estimated independently from the traffic demand itself.

For congested networks there is no known regression approach which can take into account the dependency of the route choice and the resulting interzonal travel disutilities on the traffic demand to be estimated. Ignoring this dependency produces serious internal inconsistencies in the models.

For subarea analysis, each trip is only a portion of a longer trip which originates and terminates outside the studied area. Trying to explain the causal nature of the traffic demand with respect to the pseudo origins and destinations of the subnetwork seems a futile task. Descriptive modeling seems to be a more appropriate avenue. All maximum entropy-based models reviewed in this paper share this common character. Multiproportional models are mostly applicable to uncongested networks and network equilibrium models should be used where the congestion effect produces a distorsion in route choice.

Applications of the current models are still limited. Empirical studies have concentrated essentially on sensitivity analyses of particular models. The results found are generally inconclusive, due mainly to the unreliability of the "observed" trip matrix, and imperfections in the statistical tests used. The emerging concensus is that traffic counts cannot capture all the structural particularities of the trip matrix, so that the accuracy of the resulting estimates is highly sensitive to the a priori information contained in the reference input trip matrix $\{t_{ij}\}$.

The field is wide open to challenging research. Further in-depth empirical studies with better data and more theoretical analyses should be undertaken to refine the current methodology. Open issues which need to be addressed include for example the qualitative characterization of the estimates resulting from the different models and methods, the sampling technique used in collecting traffic counts and the a priori estimation of traffic proportions.

REFERENCES

1. D. Arbour, P. Bégin, J.-P. Gohier, G. Latour et J. Thivierge, "Le modèle de transport." Textes polycopiés présentés au Comité AMUR-ETEN, Service d'urbanisme, Ville de Montréal, 1969.

2. D. Arbour, P. Bégin, J.-P. Gohier, G. Latour et J. Thivierge, "Le modèle de transport." Schéma, Service d'urbanisme, Ville de Montréal, 1971.

3. D. Arbour, P. Bégin, J.-P. Gohier, G. Latour et J. Thivierge, "Le modèle de transport: simulations métro 1981." Service d'urbanisme, Ville de Montréal, 1971.

4. M. Bacharach, "Biproportional Matrices and Input-Output Change." Cambridge University Press, 1970.

5. A. Babin, "Procédures de résolution d'un modèle de reconstitution de matrices O/D." Publication # 130, Centre de recherche sur les transports, Univeraité de Montréal, 1979.

6. D. Bear, "Principles of Telecommunication-Traffic Engineering." Peregrinus, Stevenage England, 1978.

7. Bexelius, Sten, G. Nimmerfjord, S. Nordquist and E. Read, "Studies in Traffic Genoretics." National Swedish Building Research, Doc. No. 2, UDC 656.021, 1969.

8. Y.M.M. Bishop, S.E. Fienberg and P.M. Holland, "Discrete Multivariate Analysis." MIT Press, 1975.

9. L.M. Bregman, "The relaxation Method of Finding the Common Point of Convex Sets and its Application to the Solution of Problems in Convex Programming." USSR Comp. Math. Math. Phys., Vol. 7, No. 3, pp. 200-217, 1967.

10. L. Brillouin, "Science and Information Theory." Academic Press, New York, 1956.

11. S. Debaille, "Un modèle de reconstitution de matrices origine-destination en milieu urbain." Institut de Recherche des Transports, Arcueil, France, IRT no. 27, 1977.

12. O. Desforges, "Une méthode d'enquête origine-destination par photographie aérienne: Evaluation et comparaison avec une méthode au sol." Institut de Recherche des Transports, Arcueil, France, IRT no. 18, 1976.

13. W.E. Deming and F.F. Stephan, "On a Least Squares Adjustment of a Sampled Frequency Table when the Expected Marginal Totals are Known." Annal of Mathematical Statistics, Vol. 11, pp. 427-444, 1940.

14. R.B. Dial, "Probabilistic Multipath Traffic Assignment Model which Obviates Path Enumeration." Transportation Research, Vol. 5, pp. 83-111, 1971.

15. J.-P. Dugas, "Estimation et mise à jour d'une matrice input output interregionale pour le Canada." Thèse de maîtrise, département de Sciences économiques, Université de Montréal, 1982.

16. M. Eldor, "Integrated Real-Time Freeway On-Ramp Control Strategies." Dissertation, University of California, Berkeley, Institute of Transportation Studies, 1976.

17. S. Erlander, S. Nguyen and N. Stewart, "On the Calibration of the Combined Distribution-Assignment Model." Transportation Research B, Vol. 13B, pp. 259-267, 1979.

18. A.W. Evans, "Some Properties of Trip Distribution Methods." Transportation Research, Vol. 4, pp. 19-36, 1970.

19. S.P. Evans and H.R. Kirby, "A Three-Dimensional Furness Procedure for Calibrating Gravity Models." Transportation Research, Vol. 8, pp. 105-122, 1974.

20. K.P. Furness, "Time Function Iteration." Traffic Engineering & Control, Vol. 7, No. 7, pp. 458-460, 1965.

21. M.I.J. Gaudry and L. Lamarre, "Estimating Origin-Destination Matrices from Traffic Counts: A Simple Linear Intercity Model for Quebec." The Logistic and Transportation Review, Vol. 15, No. 5, 1979.

22. R. Gordon, R. Bender and G.T. Herman, "Algebraic Reconstruction Techniques (ART) for Three Dimensional Election Microscopy and X-Ray Photography." J. Theo. Biol., Vol. 29, pp. 471-481, 1970.

23. W.M. Gorman, "Estimating Trends in Leontief Matrices." A Note on Mr. Bacharach's paper, note, Nuffield College, Oxford, duplicated 1963.

24. Y.J. Gur, M. Turnquist, M. Schneider, L. LeBlanc and D. Kurth, "Estimation of an Origin-Destination Trip Table Based on Observed Link Volumes and Turning Movements - Volume 1." Technical Report, FHWA, U.S. Department of Transportation, Washington, D.C., Report No. RD-801034, 1980.

25. Y.T. Gur, P. Hutsebaut, D. Kurth, M. Clark and J. Castilia, "Estimation of an Origin-Destination Trip Table Based on Observed Link Volumes and Turning Movements - Volume 2." User's Manual, FHWA, U.S. Department of Transportation, Washington, D.C., Report No. RD-801035, 1980.

26. A.F. Han, R.G. Dowling, E.C. Sullivan and A.D. May, "Deriving Origin-Destination Information from Routinely Collected Traffic Counts - Volume II: Trip Table Synthesis for Multipath Networks." Institute of Transportation Studies, University of California, Research Report UCB-ITS-RR-81-9, 1981.

27. E. Hauer and B.T. Shin, "Estimation of Origin-Destination Matrices from Observed Flows: Simple Systems." University of Toronto/ York University, Research Report No. 65, March 1980.

28. P. Hogberg, "Estimation of Parameters in Models for Traffic Prediction: A Non-Linear Regression Approach." Transportation Research, Vol. 10, pp. 263-265, 1976.

29. J. Holm, T. Jensen, S.K. Nielsen, A. Christensen, B. Johnsen and G. Ronby, "Calibrating Traffic Models on Traffic Census Results Only." Traffic Engineering and Control, Vol. 17, pp. 137-140, 1976.

30. C.T. Ireland and S. Kullback, "Contingency Tables with Given Marginals." Biometrika, Vol. 55, No. 1, pp. 179-188, 1968.

31. E. Jaynes, "Information Theory and Statistical Mechanics." Physical Review, Vol. 106, pp. 171-190, Vol. 108, pp. 620-630, 1957.

32. T.R. Jefferson and C.H. Scott, "The Analysis of Entropy Models with Equality and Inequality Constraints." Transportation Research, Vol. 13B, pp. 123-132, 1979.

33. K. Jornsten and S. Nguyen, "On the Estimation of a Trip Matrix from Network Data." Publication # 153, Centre de recherche sur les transports, Université de Montréal, 1979.

34. J. Kruithof, "Calculation of Telephone Traffic." De Ingenieur (E. Electrotechnik 3), Vol. 52, pp. 15-25, 1937.

35. S. Kullback, "Information Theory and Statistics." John Wiley, New York, 1959.

36. L. Lamarre, "Une méthode linéaire simple d'estimation de la matrice des origines et des destinations à partir de comptages sur les liens d'un réseau: une application au réseau routier du Québec." Publication no. 76, Centre de recherche sur les transports, Université de Montréal, 1977.

37. B. Lamond and N.F. Stewart, "Bregman's Balancing Method." Transportation Research B, Vol. 15B, No. 4, pp. 239-248, 1981.

38. W.H. Lawton and E.A. Sylvestre, "Elimination of Linear Parameters in Non-Linear Regression." Technometrics, Vol. 13, pp. 461-467, 1971.

39. L. LeBlanc and K. Farhangian, "Selection of a Trip Table which Reproduces Observed Link Flows." Southern Methodist University, Department of Operations Research and Engineering Management, Technical Report OREM 80007, April 1980.

40. A. Lent, "A Convergent Algorithm for Maximum Entropy Image Restoration, with A Medical X-Ray Application." SPSE Conference Proceedings, Toronto, Canada, 1976.

41. D. Low, "A New Approach to Transportation System Modeling." Traffic Quarterly, Vol. 26, pp. 391-404, 1972.

42. S.H. Macgill, "Convergence and Related Properties for a Modified Biproportional Problem." Environment Planning A, Vol. 11, pp. 499-506, 1979.

43. D.W. Marquardt, "An Algorithm for Least Squares Estimation of Non-Linear Parameters." J. Soc. Ind. Appl. Math, Vol. 2, pp. 431-441, 1963.

44. J.D. Murchland, "The Multiproportional Problem." Manuscript JDM-263, University College London, Transport Studies Group, 1977.

45. S. Nguyen, "Estimating an OD Matrix from Network Data: A Network Equilibrium Approach." Publication # 60, Centre de recherche sur les transports, Université de Montréal, 1977.

46. S. Nguyen, "On the Estimation of an O-D Trip Matrix by the Equilibrium Method Using Pseudo Delay Functions." Publication # 81, Centre de recherche sur les transports, Université de Montréal, 1978.

47. S. Nguyen, "A Unified Approach to Equilibrium Methods of Traffic Assignment." In Traffic Equilibrium Methods, M. Florian, ed., Vol. 118, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, Berlin, 1978.

48. S. Nguyen, "Modèles de distribution spatiale tenant compte des itinéraires." Publication # 225, Centre de recherche sur les transports, Université de Montréal, 1981.

49. N.L. Nihan, "Use of Volume Data to Reproduce Ramp-to-Ramp Freeway Trip Patterns - A Pilot Study." Washington State Department of Transportation, Research Report No. 35.1, June 1979.

50. Ovengaard, "Development of a Simplified Traffic Model from the City of Silkeborg." O.E.C.D. Seminar in Copenhagen, 1972.

51. P. Robillard and N.F. Stewart, "Iterative Numerical Methods for Trip Distribution Problems." Transportation Research, Vol. 8, pp. 575-582, 1974.

52. P. Robillard, "Estimating the OD Matrix from Observed Link Volumes." Transportation Research, Vol. 9, pp. 123-128, 1975.

53. C.E. Shannon, "A Mathematical Theory of Communication." Bell System Tech. J., Vol. 27, pp. 379-423, 623-656, 1948.

54. R. Sinkhorn, "A Relationship Between Arbitrary Positive Matrices and Doubly Stochastic Matrices." Ann. Math. Stat., Vol. 35, pp. 876-879, 1964.

55. D.P. Smith and B.G. Hutchinson, "Goodness of Fit Statistic for Trip Distribution Models." Transportation Research A, Vol. 15A, pp. 295-303, 1981.

56. R. Smock, "A Iterative Assignment Approach to Capacity Restraint on Arterial Networks." HRB, Bulletin 347, 1962.

57. F. Snickars and J.W. Weibull, "A Minimum Information Principle, Theory and Practice." Regional Science and Urban Economics, Vol. 7, pp. 137-168, 1977.

58. J. Symons, R. Wilson and J. Paterson, "A Model of Intercity Motor Travel Estimated by Link Volumes." Australian Road Research Board Proceedings, Vol. 8, 1976.

59. J. Tremblay, "Estimation d'une matrice OD basée sur les comptages du trafic en téléphone." Thèse de maîtrise, Département d'informatique et rech. opér., Université de Montréal, 1980.

60. J.H. Van Zuylen and L.G. Willumsen, "The Most Likely Trip Matrix Estimated from Traffic Counts." Transportation Research B, Vol. 14B, pp. 281-293, 1980.

61. D. Van Vliet and L.G. Willumsen, "Validation of the ME2 Model for Estimating Trip Matrices from Traffic Counts." Paper presented at the Eighth International Symposium on Transportation and Traffic Theory, University of Toronto, Toronto, Canada, June 1981.

62. J.G. Wardrop, "Some Theoretical Aspects of Road Traffic Research." Proc. Inst. Civil, Engineers, Part II, pp. 325-378, 1953.

63. A.E. Willis and A.D. May, "Deriving Origin-Destination Information from Routinely Collected Traffic Counts - Volume I: Trip Table Synthesis of Single Path Networks." Institute of Transportation Studies, University of California, Research Report UCB-ITS-RR-81-8, 1981.

64. M.J. Wills, "Development of the Highway Network, Traffic Flow and the Growth of Settlements in Interior British Columbia." M.A. Thesis, Department of Geography, The University of British Columbia, Vancouver, 1971.

65. M.J. Wills, "Spatial Linkages Submodel: Inter-Urban Passenger Movement Project: Models and Computer Programs." Ministry of State for Urban Affairs, Ottawa, 1974.

66. L. Willumsen, "Estimation of an O-D Matrix from Traffic Counts: A Review." Working Paper No. 99, Institute of Transport Studies, Leeds University, 1978.

67. A.G. Wilson, "A Statistical Theory of Spatial Distribution Models." Transportation Research, Vol. 1, pp. 253-269, 1967.

68. A.G. Wilson, "Entropy in Urban and Regional Modelling." Pion Limited, London, 1970.

69. W.T. Zangwill, "Nonlinear Programming: A Unified Approach." Prentice-Hall, Englewood Cliffs, N.J., 1969.